

MAXIMAL DIVISORIAL IDEALS AND t -MAXIMAL IDEALS

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ABSTRACT. We give conditions for a maximal divisorial ideal to be t -maximal and show with examples that, even in a completely integrally closed domain, maximal divisorial ideals need not be t -maximal.

INTRODUCTION

The v -operation and the t -operation are the the best known and most useful star operations; mainly because the structure of certain semigroups of t -ideals reflects the multiplicative properties of an integral domain. In this context an important role is played by the prime and the maximal v - and t -ideals.

Since the t -operation is a star operation of finite type, a domain R has always t -maximal ideals. On the other hand, the set of v -maximal ideals may be empty.

In this paper we deal with the following question:

Assume that M is a v -maximal ideal of R , is M necessarily a t -maximal ideal?

We show that although the answer is positive in a large class of domains, namely in the class of v -coherent domains, it is negative in general. In fact we give two examples of a v -maximal ideal P that is not a t -maximal ideal. In the first example P is an upper to zero of a completely integrally closed polynomial ring, thus P is v -invertible. In the second example P is a strongly divisorial ideal of an integrally closed semigroup ring.

1. PRELIMINARIES AND NOTATIONS

Throughout this paper R will denote an integral domain with quotient field K . We will refer to a fractional ideal as an *ideal* and will call a fractional ideal contained in R an *integral ideal*.

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We recall that a star operation is an application $I \rightarrow I^*$ from the set $F(R)$ of nonzero ideals of R to itself such that:

- (1) $R^* = R$ and $(aI)^* = aI^*$, for all $a \in K \setminus \{0\}$;
- (2) $I \subseteq I^*$ and $I \subseteq J \Rightarrow I^* \subseteq J^*$;
- (3) $I^{**} = I^*$.

General references for systems of ideals and star operations are [13, 16, 17, 22].

We denote by $f(R)$ the set of nonzero finitely generated ideals of R . A star operation $*$ is of *finite type* if, for each $I \in F(R)$, $I^* = \cup \{J^* \mid J \subseteq I \text{ and } J \in f(R)\}$. To any star operation $*$, we can associate a star operation $*_f$ of finite type by defining $I^{*f} = \cup \{J^* \mid J \subseteq I \text{ and } J \in f(R)\}$. Clearly $I^{*f} \subseteq I^*$.

The v - and the t -operations are particular star operations, defined in the following way.

For a pair of nonzero ideals I and J of a domain R we let $(J : I)$ denote the set $\{x \in K \mid xI \subseteq J\}$ and $(J :_R I)$ denote the set $\{x \in R \mid xI \subseteq J\}$. We set $I_v = (R : (R : I))$ and $I_t = \bigcup J_v$ with the union taken over all finitely generated ideals J contained in I .

The t -operation is the finite type star operation associated to the v -operation.

A nonzero ideal I is called a $*$ -ideal if $I = I^*$. Thus a nonzero ideal I is a v -ideal, or is *divisorial*, if $I = I_v$, and it is a t -ideal if $I = I_t$. Note that I is a t -ideal if and only if $J_v \subseteq I$ whenever J is finitely generated and $J \subseteq I$.

The set $F_*(R)$ of $*$ -ideals of R is a semigroup with respect to the $*$ -multiplication, defined by $(I, J) \rightarrow (IJ)^*$, with unity R .

We say that an ideal $I \in F(R)$ is $*$ -invertible if I^* is a unit in the semigroup $F_*(R)$. In this case the $*$ -inverse of I is $(R : I)$. Thus I is $*$ -invertible if and only if $(I(R : I))^* = R$. Invertible ideals are $(*$ -invertible) $*$ -ideals.

We have $I \subseteq I^* \subseteq I_v$, so that any divisorial ideal is a $*$ -ideal and any $*$ -invertible ideal is v -invertible. In particular a divisorial ideal is a t -ideal and a t -invertible ideal is v -invertible.

A nonzero ideal I is $*$ -finite if $I^* = J^*$ for some finitely generated ideal J . Since the v - and the t -operation coincide on finitely generated ideals and since $I_t = J_t$ implies $I_v = J_v$, an ideal I is t -finite if and only if $I_v = J_v$ (equivalently $(R : I) = (R : J)$) for some finitely generated ideal $J \subseteq I$. It follows that the set $f_v(R)$ of the v -finite divisorial ideals coincides with the set of the t -finite t -ideals. $f_v(R)$ is a sub-semigroup of $F_v(R)$.

An ideal I is t -invertible if and only if it is v -invertible and both I and $(R : I)$ are t -finite. Hence the set of the t -invertible t -ideals of R , here denoted by $T(R)$, is the largest subgroup of $f_v(R)$. The importance of the notion of t -invertibility is well illustrated in [28].

Denoting by $Inv(R)$ the group of the invertible ideals of R , we have

$$Inv(R) \subseteq T(R) \subseteq f_v(R) \subseteq F_v(R) \subseteq F_t(R) \subseteq F(R)$$

and

$$Inv(R) \subseteq f(R).$$

Several important classes of domains may be characterized by the fact that some of these inclusions are equalities. For example R is a *Prüfer* domain if and only if $Inv(R) = f(R)$ [13], it is a *Krull* domain if and only if $T(R) = F_t(R)$ [24] and it is a *Prüfer v -multiplication* domain, for short a *PvMD*, if and only if $T(R) = f_v(R)$ [16]. A *Mori* domain is a domain satisfying the ascending chain condition on integral divisorial ideals and has the property that $f_v(R) = F_t(R)$. Noetherian and Krull domains are Mori. A recent reference for Mori domains is [1]. The class of domains with the property that $F_v(R) = F(R)$ have been studied by several authors [3, 18, 26, 4, 5]. A domain such that $F_v(R) = F_t(R)$ is called in [20] a *TV-domain*. Examples of *TV*-domains are given in [20, 23, 7]. Mori and pseudovaluations domains which are not valuation domains are *TV*-domains.

2. WHEN A MAXIMAL DIVISORIAL IDEAL IS t -MAXIMAL

A prime $*$ -ideal is called a $*$ -prime. A $*$ -maximal ideal is an ideal that is maximal in the set of the proper integral $*$ -ideals. A v -maximal ideal is also called a *maximal divisorial ideal*. A $*$ -maximal ideal is a prime ideal (if it exists).

If $*$ is a star operation of finite type, an easy application of the Zorn Lemma shows that the set $*\text{-Max}(R)$ of the $*$ -maximal ideals of R is not empty. Moreover, for each $I \in F(R)$, $I^* = \bigcap_{M \in *\text{-Max}(R)} I^* R_M$ [16]. In particular the set of the t -maximal ideals is not empty and $I_t = \bigcap_{M \in t\text{-Max}(R)} I_t R_M$. On the contrary, the set of maximal divisorial ideals may be empty, like for example when R is a rank-one nondiscrete valuation domain.

If M is a $*$ -maximal ideal that is not $*$ -invertible, then $M = (M(R : M))^*$ and so $(M : M) = (R : M)$. An ideal I with the property that $(R : I) = (I : I)$ is called *strong*. A strong ideal is never $*$ -invertible and we have just seen that a $*$ -maximal ideal is either $*$ -invertible or strong.

An ideal which is strong and divisorial is called *strongly divisorial*.

Proposition 2.1.

- (1) *If M is a maximal divisorial ideal of R , then $M = x^{-1}R \cap R$, for some element $x \in K$. Hence $(R : M) = (R + xR)_v$.*
- (2) *If P is a prime divisorial ideal of R such that $(R : P) = R + xR$, for some element $x \in K$, then P is maximal divisorial.*

Proof. (1) If $x \in (R : M) \setminus R$, then $M \subseteq x^{-1}R$ and $R \not\subseteq x^{-1}R$. Since an intersection of divisorial ideals is divisorial and M is v -maximal, we have $M = x^{-1}R \cap R = (R : R + xR)$.

(2) Let Q be a proper divisorial ideal containing P . Since Q is divisorial, $(R : Q) \not\subseteq R$. Since $(R : Q) \subseteq (R : P) = R + xR$, we see that there exists an element $y \in R$ such that $xy \in (R : Q) \setminus R$. Thus $y \notin P$, and $xyQ \subseteq R$. Since $P = (R : R + xR)$, we obtain that $yQ \subseteq P$. Since P is a prime ideal, we conclude that $Q \subseteq P$. Hence P is maximal divisorial. \square

In a Mori domain R , all the prime divisorial ideals are of the form $x^{-1}R \cap R = (R : R + xR)$ [19, Corollary 2.5].

A domain has the property that each t -maximal ideal is divisorial if and only if every ideal I such that $(R : I) = R$ is t -finite [20, Proposition 2.4]. A domain of this type is called an H -domain in [15]. A TV -domain is clearly an H -domain, but the converse is not true [20, 2].

The following proposition gives conditions for a divisorial prime ideal to be a t -maximal ideal. A proof can be found in [10].

Proposition 2.2.

- (1) *A v -invertible divisorial prime is maximal divisorial;*
- (2) *A v -finite maximal divisorial ideal is t -maximal;*
- (3) *A v -finite v -invertible divisorial prime is t -invertible;*
- (4) *A t -invertible t -prime is t -maximal.*

We remark that in general a $*$ -invertible $*$ -prime need not be $*$ -maximal (for example a principal prime ideal is not necessarily a maximal ideal) and that a (non-prime) v -finite v -invertible divisorial ideal need not be t -invertible [6].

Corollary 2.3. *Assume that each maximal divisorial ideal of R is a t -maximal ideal. Then each v -invertible divisorial prime is a t -invertible t -maximal ideal.*

Proof. Let P be a v -invertible divisorial prime. By Proposition 2.2, P is maximal divisorial and so t -maximal. Since P is not strong, then it is t -invertible. \square

In general, if each v -invertible divisorial prime of R is t -invertible, it is not true that each v -invertible ideal is t -invertible. This last property

is in fact equivalent to R being an H -domain [28, Proposition 4.2]. The ring of entire functions is not an H -domain, but all its divisorial primes are t -invertible (see for example [10, Section 2]).

A v -coherent domain is a domain R with the property that, for each finitely generated ideal J , the ideal $(R : J)$ is v -finite. This class of domains was first studied (under a different name) in [25] and is very large, properly including PvMD's, Mori domains and coherent domains [25, 11]. (A domain is *coherent* if each finitely generated ideal is finitely presented, or, equivalently, if the intersection of each pair of finitely generated ideals is finitely generated.)

Proposition 2.4. *If R is v -coherent, then each maximal divisorial ideal is t -maximal.*

Proof. Let M be a maximal divisorial ideal of R . Then $M = x^{-1}R \cap R = (R : R + xR)$ for some $x \in K$ (Proposition 2.1). Since R is v -coherent, then M is v -finite and so t -maximal by Proposition 2.2. \square

A domain R is *completely integrally closed* if and only if $F_v(R)$ is a group under v -multiplication [13]. If $F_v(R) = T(R)$, then R is a completely integrally closed H -domain, equivalently a Krull domain [10, 15].

A divisorial prime of a completely integrally closed domain, being v -invertible, is always maximal divisorial by Proposition 2.2. We will see in the next section that it need not be t -maximal. As a matter of fact, a divisorial prime P of a completely integrally closed domain has height one and P is t -maximal if and only if it is v -finite, if and only if it is t -invertible [10, Theorem 2.3].

A completely integrally closed v -coherent domain is a (completely integrally closed) PvMD. In this case each divisorial prime is t -maximal by Corollary 2.3.

We now turn to the case of polynomial rings.

We denote by \mathbf{X} a set of independent indeterminates over R and by $R[\mathbf{X}]$ the polynomial ring in this set of indeterminates.

It is well known that the correspondence $I \mapsto I[\mathbf{X}]$ induces inclusion preserving injective maps $t(R) \rightarrow t(R[\mathbf{X}])$ and $D(R) \rightarrow D(R[\mathbf{X}])$. Moreover, M is a t -maximal ideal, respectively a maximal divisorial ideal, of $R[\mathbf{X}]$ such that $M \cap R \neq (0)$, if and only if $M = (M \cap R)[\mathbf{X}]$ and $M \cap R$ is a t -maximal ideal, respectively a maximal divisorial ideal, of R (see for example [8, Lemma 2.1] and [27, Theorem 3.6]).

Thus, if each maximal divisorial ideal of $R[\mathbf{X}]$ is t -maximal, R has the same property.

On the other hand, Example 3.1 in the next section shows that if $M \cap R = (0)$, then M may be maximal divisorial but not t -maximal.

A prime ideal Q of $R[\mathbf{X}]$ such that $Q \cap R = (0)$ is called an *upper to zero*. Q is an upper to zero of height one if and only if $Q = fK[\mathbf{X}] \cap R[\mathbf{X}]$ for some polynomial $f \in R[\mathbf{X}]$, irreducible in $K[\mathbf{X}]$ [12, Lemma 2.1]. In one indeterminate, all the uppers to zero are of this form.

Recall that if R is integrally closed and f is a nonzero polynomial of $R[\mathbf{X}]$, then $fK[\mathbf{X}] \cap R[\mathbf{X}] = f(R : c(f))[\mathbf{X}]$ [13, Corollary 34.9]. (Here $c(f)$ denotes the *content* of f , that is the fractional ideal of R generated by the coefficients of f .) Hence if R is integrally closed, an upper to zero of height one is always divisorial and if R is completely integrally closed, an upper to zero of height one, being v -invertible, is always maximal divisorial.

In general, an upper to zero is t -maximal if and only if it is t -invertible; in this case it has height one [12, Section 3]. We now show that a similar result holds for the v -operation.

Proposition 2.5. *A divisorial upper to zero is a maximal divisorial ideal if and only if it is v -invertible. In this case it has height one.*

Proof. A divisorial v -invertible prime is always maximal divisorial (Proposition 2.2 (1)).

Conversely, let $P \subseteq R[\mathbf{X}]$ be an upper to zero that is maximal divisorial. Then $P = \frac{f}{g}R[\mathbf{X}] \cap R[\mathbf{X}] \subseteq fK[\mathbf{X}] \cap R[\mathbf{X}]$, for some $f, g \in R[\mathbf{X}]$, $g \neq 0$ (Proposition 2.1(1)). Since $P \cap R = (0)$ and $f = \frac{f}{g}g \in P$, then $f \notin R$. We may also assume that f and g are coprime in $K[\mathbf{X}]$.

Let $h = f\alpha \in fK[\mathbf{X}] \cap R[\mathbf{X}]$, with $\alpha \in K[\mathbf{X}]$. There is a nonzero $c \in R$ such that $c\alpha \in R[\mathbf{X}]$. Hence $ch = (c\alpha)f = (c\alpha g)\frac{f}{g} \in P$. Since $c \notin P$, then $h \in P$.

We conclude that $P = fK[\mathbf{X}] \cap R[\mathbf{X}]$ has height one.

In addition, $\frac{g}{f} \in (R[\mathbf{X}] : P)$, but $\frac{g}{f} \notin (P : P)$. Otherwise $g = \frac{g}{f}f \in P$ and so $g = \frac{f}{g}t$ for some $t \in R[\mathbf{X}]$. Then f divides g^2 in $K[\mathbf{X}]$, which is impossible, because f and g are coprime and $f \notin K$.

It follows that P is not strong and, being maximal divisorial, is v -invertible. \square

The following result was proved in [15] for one indeterminate.

Proposition 2.6. *R is an H -domain if and only if $R[\mathbf{X}]$ is an H -domain.*

Proof. An extended prime $P[\mathbf{X}]$ is a t -maximal ideal, respectively a maximal divisorial ideal, if and only if so is P , [8, Lemma 2.1] and

[27, Theorem 3.6]. A t -maximal upper to zero is t -invertible by [12, Theorem 2.3]. Hence it is divisorial. \square

The domain R is said to be a *UMT-domain* if every upper to zero of $R[X]$ is a t -maximal ideal [21]. This property is stable under polynomial extensions, in fact R is a *UMT-domain* if and only if $R[\mathbf{X}]$ is a *UMT-domain* [8, Theorem 2.4]. The integrally closed *UMT*-domains are exactly the PvMDs [21, Proposition 3.2].

The following proposition is immediate.

Proposition 2.7. *Assume that R is an UMT-domain. Then each maximal divisorial ideal of R is t -maximal if and only if $R[\mathbf{X}]$ has the same property.*

We conclude this section recalling that it is not known whether R v -coherent implies that $R[X]$ is v -coherent. This is true under the additional hypothesis that R is integrally closed [25]. In this case, each prime of $R[X]$ upper to zero is divisorial v -finite. When R is v -coherent and completely integrally closed (thus a completely integrally closed PvMD), each upper to zero of $R[X]$ is t -maximal (and t -invertible).

3. MAXIMAL DIVISORIAL IDEALS THAT ARE NOT t -MAXIMAL

In this section we give two examples of a maximal divisorial ideal P of an integral domain R that is not a t -maximal ideal. In the first example R is a completely integrally closed polynomial ring in one indeterminate and P is an upper to zero, thus P is v -invertible. In the second example R is an integrally closed semigroup ring and P is strongly divisorial.

Example 3.1. *An upper to zero P of a completely integrally closed polynomial ring $R[X]$ that is maximal divisorial but not t -maximal. P is necessarily v -invertible.*

Let y, z and $\mathbf{t} = \{t_n(n \geq 1)\}$ be independent indeterminates over a field k . Let S be the semigroup of monomials f of $k[y, z, \mathbf{t}]$ satisfying the conditions $\deg_{y,z} f \geq \deg_{t_n} f$ for all $n \geq 1$, and let $R = k[S]$ the semigroup ring over k generated by S .

Set

$$P = (y + zX)K[X] \cap R[X],$$

where K is the field of fractions of R and X is an indeterminate over R . Then R (and so also $R[X]$) is completely integrally closed, and P is a maximal divisorial ideal of $R[X]$ that is not t -maximal.

Proof.

- (1) $R[X]$ is completely integrally closed.

It is enough to show that R is completely integrally closed. Since $R = k[S]$ is a semigroup ring over the field k , by [14, Corollary 12.7 (2)] to this end it suffices to show that the semigroup S is completely integrally closed.

Let $u, v, w \in S$ so that $u(\frac{v}{w})^m \in S$ for all $m \geq 1$. Fix $n \geq 1$. Then $\deg_{y,z}(u(\frac{v}{w})^m) \geq \deg_{t_n}(u(\frac{v}{w})^m)$ for all m . Hence

$$\deg_{y,z} u + m \deg_{y,z}(\frac{v}{w}) \geq \deg_{t_n} u + m \deg_{t_n}(\frac{v}{w}).$$

Divide by m and let m go to ∞ to obtain that $\deg_{y,z}(\frac{v}{w}) \geq \deg_{t_n}(\frac{v}{w})$. The same argument shows that $\frac{v}{w}$ is a monomial, that is has a nonnegative degree in each indeterminate. It follows that $\frac{v}{w} \in S$; thus S is completely integrally closed.

- (2) P is an upper to zero of $R[X]$ that is a v -invertible maximal divisorial ideal.

P is clearly an upper to zero. Since R is integrally closed, then $P = (y + zX)(R : (y, z))[X]$ by [13, Corollary 34.9], hence P is divisorial. But $R[X]$ is completely integrally closed; thus P is v -invertible and so maximal divisorial (Proposition 2.2).

- (3) P is not t -maximal.

Let $Q = (y, z)k[y, z, \mathbf{t}] \cap R$. Then $QR[X]$ is a proper t -ideal of $R[X]$ properly containing P .

To verify this, let F be a finite subset of $QR[X]$. Let t_n be an indeterminate that does not occur in the polynomials in the set F . Then $t_n f \in R[X]$ for all $f \in F$, so $t_n \in (R[X] : F)$. If $g \in (F)_v$, then $gt_n \in R[X]$. Hence $\deg_{y,z} gt_n \geq 1$ and $g \in Q$. It follows that $(F)_v \subseteq Q$, so Q is a t -ideal. □

Example 3.2. *An example of a strong maximal divisorial ideal of an integrally closed domain R that is not t -maximal.*

Let k be a field and let $Y, Z, \mathbf{X} = \{X_n : n \geq 1\}, \mathbf{T} = \{T_n : n \geq 1\}$ be independent indeterminates over k . Let S be the set of monomials f in $k[Y, Z, \mathbf{X}, \mathbf{T}]$ satisfying the following two conditions:

- (a) If Z occurs in f , then some X_n occurs in f .
- (b) For all n , if T_n occurs in f , then either Y or X_i occurs in f for some $i \leq n$.

Clearly, S is a semigroup containing \mathbf{X} and Y . Let $R = k[S]$ be the semigroup ring over S and set

$$P = (\mathbf{X})k[Y, Z, \mathbf{X}, \mathbf{T}] \cap R.$$

Then R is integrally closed and P is a strong maximal divisorial ideal of R that is not t -maximal.

Proof. We will use repeatedly that P is a monomial ideal of R .

- (1) R is integrally closed.

By [14, Corollary 12.11 (2)], it is enough to show that the monoid S is integrally closed. If f is an element in the quotient group of S such that $f^n \in S$ for some $n \geq 1$, then f is a monomial. Since f^n satisfies conditions (a)-(b), it is clear that f also satisfies them, thus $f \in S$. We conclude that R is integrally closed.

- (2) $P = RZ^{-1} \cap R$. Hence P is a divisorial ideal.

Clearly, any monomial in ZP satisfies conditions (a)-(b), hence $ZP \subseteq R$. Thus $P \subseteq RZ^{-1} \cap R$.

For the reverse inclusion, it is enough to show that any monomial $f \in RZ^{-1}$ belongs to P . Since $Zf \in R$, we see that Zf satisfies conditions (a)-(b) and so does f , thus $f \in R$. Using again that $Zf \in R$ we see that some X_n occurs in f , hence $f \in P$.

- (3) $(R : P) = R[Z]$.

Using conditions (a)-(b), we see that $R[Z] \subseteq (R : P)$.

For the reverse inclusion, let u be a quotient of monomials in $(R : P)$. Since $uX_1, uX_2 \in R$, we see that uX_1 and uX_2 are monomials, hence, by factoriality, u also is a monomial. Let $u = Z^k u_0$, where $k \geq 0$, u_0 is a monomial and Z does not occur in u_0 . Choose a positive integer N such that $N > i$ for all T_i 's occurring in u . Since $Z^k u_0 X_N \in R$, we see that u_0 satisfies condition (b); hence $u_0 \in R$, so $u \in R[Z]$.

- (4) P is a strong maximal divisorial ideal.

We have $(R : P) = R[Z] \subseteq (P : P)$, thus $(R : P) = (P : P)$, that is, P is strong.

Assume that P is not maximal divisorial, so there is a divisorial ideal Q properly containing P . Let $f \in Q \setminus P$. We may assume that no X_n occurs in f , thus Z does not occur in f either by condition (a) above. Let $g \in (R : Q) \setminus R$, thus $g \in (R : P) = R[Z]$, $g = \sum_{i=0}^n a_i Z^i$, where $a_0, \dots, a_n \in R$. We may assume that $a_n Z^n \notin R$, thus $n \geq 1$. We also may assume that no X_i occurs in a_n . Thus no X_i occurs in fa_n , which implies that $fa_n Z^n \notin R$. Since $R = k[S]$, we obtain that $fg = fa_n Z^n + \dots \notin R$, a contradiction.

- (5) The ideal $M = (S)R$ is a maximal ideal of R properly containing P and is a t -ideal.

Clearly M is a maximal ideal containing P . Since $Y \in M \setminus P$, we have $P \subsetneq M$.

To show that M is a t -ideal, let F be a finite subset of M and let N be a positive integer such that $N > i$ for each T_i occurring in some element of F . From conditions (a)-(b) it follows that $M \subseteq (\mathbf{X}, Y)k[Y, Z, \mathbf{X}, \mathbf{T}]$. Hence $T_N F \subseteq R$. Thus $(F)_v \subseteq (R : T_N) \cap R$. Since $T_N \notin R$ and since $(R : T_N) \cap R$ is a monomial ideal, we obtain that $(R : T_N) \cap R \subseteq (S)R = M$. It follows that $(F)_v \subseteq M$ and that M is a t -ideal.

□

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